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ON THE UNSTEADY THREE-DIMENSIONAL BOUNDARY LAYER FREELY INTERACTING WITH THE EXTERNAL STREAM*

O. S. RYZHOV

Asymptotic equations that define unsteady processes in a three-dimensional boundary layer with self-induced pressure are derived. The pressure gradient under conditions of free interaction is, as usually, calculated not by the solution of the external problem of flow over a body, but on the assumption that it is due to growth of streamline displacement thickness near the body surface. Besides the principal terms, terms of second order of smallness are retained in asymptotic sequencies. If the characteristic dimensions of the free interaction region are the same in all directions in the plane tangent to the body surface, the system of equations defining the thin layer next to the wall must be integrated together with the system which defines the nonviscous stream.

1. The external stream. We assume that under conditions of free interaction between an unsteady three-dimensional boundary layer and the external stream three regions with essentially different properties are formed, as happens in plane-parallel flows. As the basis of our mathematical analysis we take the concepts of the nonlinear theory of perturbations first formulated in connection with the investigation of steady separation /1—6/, and subsequently extended for application to processes with time dependent parameters /7—11/. According to that theory the effects of viscosity and thermal conductivity are small and there are no vortices in the upper region 1. The effect of dissipative factors can also be neglected in the middle region 2, although the velocity field is essentially turbulent. In region 3, the thin layer next to the wall, viscosity always plays the predominant part in the formation of flow, while the effect of thermal conductivity is secondary, provided the gas temperature varies within fairly narrow limits and, consequently its compressibility virtually does not manifest itself.

We use the notation: t for time, x, y, z for Cartesian coordinates, v_x, v_y, and v_z for velocity components along these axes, ρ for density, p for pressure, and $\lambda^{(g)}$ for the first viscosity coefficient. Parameters of the unperturbed gas are denoted by the subscript ∞. We assume for simplicity that the gas flows along a plate at velocity U_∞ and the Mach number M_∞ differs from unity by a finite magnitude. We introduce the small parameter $\varepsilon = \mathrm{Re}^{-1/8}$, with the Reynolds number Re calculated with the use of the first viscosity coefficient and distance L from the plate leading edge. We locate the axes x and z in the plane subjected to flow with the x-axis coinciding with the velocity vector of the stream flowing from infinity.

We begin by analyzing the external region 1 where the flow is laminar. Assuming the importance of all Cartesian coordinates to be equivalent, we set here

$$t = L/U_\infty\,(t_0 + \varepsilon^2 t_1),\ x = L\,(1 + \varepsilon^3 x_1),\ y = \varepsilon^3 L y_1,\ z = \varepsilon^3 L z_1 \tag{1.1}$$

and expand the unknown functions in asymptotic series

$$\begin{aligned} &v_x = U_\infty\,(1 + \varepsilon^2 u_{11} + \varepsilon^3 u_{12} + \ldots),\ v_y = U_\infty\,(\varepsilon^2 v_{11} + \varepsilon^3 v_{12} + \ldots),\\ &v_z = U_\infty\,(\varepsilon^2 w_{11} + \varepsilon^3 w_{12} + \ldots),\ \rho = \rho_\infty\,(1 + \varepsilon^2 \rho_{11} + \varepsilon^3 \rho_{12} + \ldots),\\ &p = p_\infty + \rho_\infty U_\infty^2(\varepsilon^2 p_{11} + \varepsilon^3 p_{12} + \ldots) \end{aligned} \tag{1.2}$$

where t_1, x_1, y_1, z_1 are the arguments of functions $u_{1i}, v_{1i}, w_{1i}, \rho_{1i}, p_{1i}$ $(i = 1, 2, \ldots)$.

We substitute formulas (1.1) and expansions (1.2) into the system of Navier—Stokes equations and collect terms with like powers of ε. For the first approximation functions we obtain

$$\frac{\partial \rho_{11}}{\partial x_1} + \frac{\partial u_{11}}{\partial x_1} + \frac{\partial v_{11}}{\partial y_1} + \frac{\partial w_{11}}{\partial z_1} = 0,\quad \frac{\partial u_{11}}{\partial x_1} + \frac{\partial p_{11}}{\partial x_1} = 0,\quad \frac{\partial v_{11}}{\partial x_1} + \frac{\partial p_{11}}{\partial y_1} = 0,\quad \frac{\partial w_{11}}{\partial x_1} + \frac{\partial p_{11}}{\partial z_1} = 0,\quad M_\infty^2 \frac{\partial p_{11}}{\partial x_1} - \frac{\partial \rho_{11}}{\partial x_1} = 0 \tag{1.3}$$

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It is important that all equations of system (1.3) do not contain derivatives with respect to time. This means that the external inviscid stream is inert, momentarily adjusting itself to the perturbations which usually occur in problems on free interaction in region 3 close to the wall. The second approximation functions satisfy the nonhomogeneous system of linear equations

$$\frac{\partial \rho_{12}}{\partial x_1} + \frac{\partial u_{12}}{\partial x_1} + \frac{\partial v_{12}}{\partial y_1} + \frac{\partial w_{12}}{\partial z_1} = -\frac{\partial \rho_{11}}{\partial t_1}, \quad \frac{\partial u_{12}}{\partial x_1} + \frac{\partial p_{12}}{\partial x_1} = -\frac{\partial u_{11}}{\partial t_1}, \quad \frac{\partial v_{12}}{\partial x_1} + \frac{\partial p_{12}}{\partial y_1} = -\frac{\partial v_{11}}{\partial t_1}, \quad \frac{\partial w_{12}}{\partial x_1} + \frac{\partial p_{12}}{\partial z_1} = -\frac{\partial w_{11}}{\partial t_1}, \tag{1.4}$$

$$M_\infty^2 \frac{\partial p_{12}}{\partial x_1} - \frac{\partial \rho_{12}}{\partial x_1} = 0$$

The homogeneous system corresponding to it coincides with system (1.3). Time appears only in the right-hand sides of Eqs.(1.4) in whose solutions it is also contained as a parameter.

Let us partly integrate the system of Eqs.(1.3) and (1.4), assuming that all unknown functions tend to vanish as $x_1 \to -\infty,\ t_1, y_1, z_1 = \text{const}$ and $y_1 \to +\infty,\ t_1, x_1, z_1 = \text{const}$. The first of these systems yields

$$(1 - M_\infty^2)\frac{\partial^2 p_{11}}{\partial x_1^2} + \frac{\partial^2 p_{11}}{\partial y_1^2} + \frac{\partial^2 p_{11}}{\partial z_1^2} = 0, \quad u_{11} = -p_{11}, \quad \rho_{11} = M_\infty^2 p_{11} \tag{1.5}$$

$$v_{11} = -\frac{\partial}{\partial y_1}\int_{-\infty}^{x_1} p_{11}(t_1, \xi, y_1, z_1)\, d\xi, \quad w_{11} = -\frac{\partial}{\partial z_1}\int_{-\infty}^{x_1} p_{11}(t_1, \xi, y_1, z_1)\, d\xi$$

For a plane-parallel supersonic stream with $\partial/\partial z_1 = 0$ and $M_\infty > 1$ the general solution of the wave equation is determined by the Delambre's formula. From this follows the relation

$$p_{11}(t_1, x_1, 0) = (M_\infty^2 - 1)^{-1/2}\, v_{11}(t_1, x_1, 0) \tag{1.6}$$

between the perturbed pressure and the transverse component of the velocity vector when $y_1 = 0$. For a plane-parallel subsonic stream with $\partial/\partial z_1 = 0$ and $M_\infty < 1$ we obtain Neuman's problem for the Laplace equation whose solution is sought in the half-plane $y_1 > 0$. In that case we have

$$p_{11}(t_1, x_1, 0) = -\frac{1}{\pi}(1 - M_\infty^2)^{-1/2}\int_{-\infty}^{+\infty} \frac{v_{11}(t_1, \xi, 0)}{x_1 - \xi}\, d\xi \tag{1.7}$$

When the thermodynamic functions and the velocity field depend on all three space variables x, y, z, formulas (1.6) and (1.7) are no longer valid. As will become clear subsequently, the absence of simple expressions for the parameters of gas in the external stream makes it impossible to formulate separately the boundary value problem for the region next to the wall of the three-dimensional boundary layer.

The result of partial integration of the system of equations for second approximation functions shows that

$$(1 - M_\infty^2)\frac{\partial^2 p_{12}}{\partial x_1^2} + \frac{\partial^2 p_{12}}{\partial y_1^2} + \frac{\partial^2 p_{12}}{\partial z_1^2} = 2M_\infty^2 \frac{\partial^2 p_{11}}{\partial t_1\, \partial x_1}, \quad \rho_{12} = M_\infty^2 p_{12}, \quad u_{12} = -p_{12} + \frac{\partial}{\partial t_1}\int_{-\infty}^{x_1} p_{11}\, d\xi \tag{1.8}$$

$$v_{12} = -\frac{\partial}{\partial y_1}\int_{-\infty}^{x_1} p_{12}\, d\xi + \frac{\partial^2}{\partial t_1\, \partial y_1}\int_{-\infty}^{x_1}\int_{-\infty}^{\zeta} p_{11}\, d\xi\, d\zeta, \quad w_{12} = -\frac{\partial}{\partial z_1}\int_{-\infty}^{x_1} p_{12}\, d\xi + \frac{\partial^2}{\partial t_1\, \partial z_1}\int_{-\infty}^{x_1}\int_{-\infty}^{\zeta} p_{11}\, d\xi\, d\zeta$$

where t_1, ξ, y_1, z_1 are the arguments of functions p_{11}, p_{12} of the integrands.

If a plane-parallel stream is supersonic, both functions p_{12} and v_{12} are solutions of wave equations with a right-hand side. For a plane parallel subsonic stream it is possible to formulate Neuman's problem for the Poisson's equation whose solution is to be determined in the upper half-plane $y_1 > 0$. In both cases formulas linking $p_{12}(t_1, x_1, 0)$ with $v_{12}(t_1, x_1, 0)$ and first approximation functions on line $y_1 = 0$ exist /11/. But such relations do not exist for the three-dimensional boundary layer.

2. The intermediate region.

We pass to the investigation of region 2 which constitutes the basic region of the boundary layer. In spite of possibility of neglecting viscous stresses and the heat flux, the velocity field in this region contains vortices already in the first approximation. The scales of time and coordinates are specified by formulas

$$t = L/U_\infty\,(t_0 + \varepsilon^2 t_2), \quad x = L(1 + \varepsilon^3 x_2), \quad y = \varepsilon^4 L y_2, \quad z = \varepsilon^3 L z_2 \tag{2.1}$$

Expansions

$$v_x = U_\infty(U_0(y_2) + \varepsilon u_{21} + \varepsilon^2 u_{22} + \ldots), \quad v_y = U_\infty(\varepsilon^2 v_{21} + \varepsilon^3 v_{22} + \ldots) \tag{2.2}$$

$$v_z = U_\infty (\varepsilon^2 w_{21} + \varepsilon^3 w_{22} + \ldots),\ \rho = \rho_\infty (R_0(y_2) - \varepsilon\rho_{21} + \varepsilon^2\rho_{22} + \ldots),\quad p = p_\infty + \rho_\infty U_\infty^2 (\varepsilon^2 p_{21} + \varepsilon^3 p_{22} + \ldots)$$

are valid for the parameters of gas. The arguments of functions $u_{2i}, v_{2i}, w_{2i}, \rho_{2i}, p_{2i}$ $(i = 1, 2 \ldots)$ are t_2, x_2, y_2, z_2.

The comparison of formulas (1.1) and (2.1) indicates, first of all, that $t_1 = t_2$, $x_1 = x_2$ and $z_1 = z_2$, but $y_1 \neq y_2$. The importance of Cartesian coordinates in region 2 is no longer equitable, since the chracteristic length in the direction normal to the plate has been selected equal to the thickness of the unperturbed boundary layer. Structure of the latter is obtained from the Blasius solution /12/ merging with which for $x_2 \to -\infty, t_2, y_2,$ $z_2 = \text{const}$ enables us to establish the form of functions $U_0(y_2)$ and $R_0(y_2)$. Note that transverse and lateral components v_y and v_z of the velocity vector, respectively, are of comparable magnitude, and the perturbations of longitudinal velocity components considerably exceed them in the order of their amplitude.

The substitution of formulas (2.1) together with expansions (2.2) in the system of Navier—Stokes equations yields for the principal terms

$$R_0\frac{\partial u_{21}}{\partial x_2} + U_0\frac{\partial \rho_{21}}{\partial x_2} + R_0\frac{\partial v_{21}}{\partial y_2} + v_{21}\frac{dR_0}{dy_2} = 0,\qquad U_0\frac{\partial u_{21}}{\partial x_2} + v_{21}\frac{dU_0}{dy_2} = 0,\quad \frac{\partial p_{21}}{\partial y_2} = 0 \tag{2.3}$$

$$R_0U_0\frac{\partial w_{21}}{\partial x_2} + \frac{\partial p_{21}}{\partial z_2} = 0,\quad U_0\frac{\partial \rho_{21}}{\partial x_2} + v_{21}\frac{dR_0}{dy_2} = 0$$

There are again no derivatives with respect to time in all equations of system (2.3). In the first approximation oscillations in the basic part of the boundary layer are instantaneously transmitted from point to point. Only in the thin layer next to the wall can the flow have an essentially unstable character.

The fourth equation of system (2.3) is separated from the remaining which are integrated independently of it. The integral of the fourth of Eqs.(2.3) is determined after a solution is obtained for functions u_{21}, v_{21}, ρ_{21} and p_{21}. That solution defines the structure of the plane-parallel stream. This enables us to conclude that it is possible to superpose perturbations in the sideways direction on any two-dimensional boundary in the plane tangent to the surface of the body, without any disturbance in the fields of other gas parameters. The three-dimensional boundary layer in region 2 differs from the two-dimensional one only by the presence of a velocity component in the direction of the z-axis, which is determined by the pressure distribution.

For the correction terms in expansion (2.2) we obtain the nonhomogeneous system of equations

$$R_0\frac{\partial u_{22}}{\partial x_2} + U_0\frac{\partial \rho_{22}}{\partial x_2} + R_0\frac{\partial v_{22}}{\partial y_2} + v_{22}\frac{dR_0}{dy_2} = -\frac{\partial \rho_{21}}{\partial t_2} - \frac{\partial \rho_{21}u_{21}}{\partial x_2} - \frac{\partial \rho_{21}v_{21}}{\partial y_2} - R_0\frac{\partial w_{21}}{\partial z_2} \tag{2.4}$$

$$R_0U_0\frac{\partial u_{22}}{\partial x_2} + R_0v_{22}\frac{dU_0}{dy_2} = -R_0\frac{\partial u_{21}}{\partial t_2} - \frac{\partial p_{21}}{\partial x_2} - R_0u_{21}\frac{\partial u_{21}}{\partial x_2} - R_0v_{21}\frac{\partial u_{21}}{\partial y_2},\quad \frac{\partial p_{22}}{\partial y_2} = -R_0U_0\frac{\partial v_{21}}{\partial x_2}$$

$$R_0U_0\frac{\partial w_{22}}{\partial x_2} + \frac{\partial p_{22}}{\partial z_2} = -R_0\frac{\partial w_{21}}{\partial t_2} - (R_0u_{21} + U_0\rho_{21})\frac{\partial w_{21}}{\partial x_2} - R_0v_{21}\frac{\partial w_{21}}{\partial y_2}$$

$$U_0\frac{\partial \rho_{22}}{\partial x_2} + v_{22}\frac{dR_0}{dy_2} = -\frac{\partial \rho_{21}}{\partial t_2} - u_{21}\frac{\partial \rho_{21}}{\partial x_2} + M_\infty^2 R_0U_0\frac{\partial p_{21}}{\partial x_2} - v_{21}\frac{\partial \rho_{21}}{\partial y_2}$$

The corresponding homogeneous system, although of the form of system (2.3), is linear. Time appears only in the right-hand sides of Eqs.(2.4), hence in their solutions it represents a parameter. The parametric time dependence is thus distinctive feature of expansions that define the perturbed stream field in the upper region 1, as well as in the intermediate region 2.

The fourth of Eqs.(2.4) can be separated from the remaining which constitute a closed system which differs from that corresponding to the two-dimensional boundary layer only by the term $-R_0\partial w_{21}/\partial z_2$ in the right-hand side of the first of its equations.

Passing to the integration of the last two systems of equations, we stipulate the damping of perturbations in region 2 at infinity upstream of the flow. For the principal terms we have the explicit formulas

$$u_{21} = A_1\frac{dU_0}{dy_2},\quad v_{21} = -\frac{\partial A_1}{\partial x_2}U_0(y_2),\quad \rho_{21} = A_1\frac{dR_0}{dy_2}, \tag{2.5}$$

$$p_{21} = p_{21}(t_2, x_2, z_2),\quad w_{21} = -\frac{1}{R_0(y_2)U_0(y_2)}\frac{\partial}{\partial z_2}\int_{-\infty}^{x_2} p_{21}(t_2, \xi, z_2)\,d\xi$$

The arbitrary function $A_1(t_2, x_2, z_2)$ satisfies the condition $A_1 \to 0$ for $x_2 \to -\infty$, t_2, $z_2 = \text{const}$. The meaning of this simple solution is that the streamline in the boundary layer is displaced, and at each cross section $z_2 = \mathrm{const}$ the momentary magnitude of the latter in the two-dimensional stream, which is the basis for the construction the three-dimensional velocity field, is obtained by the substitution of $y_2 + \varepsilon A_1(t_2, x_2, z_2)$ for y_2 in the Blasius solution.

The system of Eqs.(2.4) can be partly integrated. Taking into account the explicit form of solution for first approximation functions, we obtain

$$p_{22} = p_{22}(t_2, x_2, 0, z_2) + \left[y_2 - \int_0^{y_2} \frac{M_\infty^2 - M_0^2(\eta)}{M_\infty^2} d\eta \right] \frac{\partial^2 A_1}{\partial x_2^2}, \quad M_0^2(y_2) = M_\infty^2 R_0(y_2) U_0^2(y_2) \tag{2.6}$$

$$v_{22} = -\frac{\partial A_1}{\partial t_2} - U_0 \frac{\partial A_2}{\partial x_2} - A_1 \frac{\partial A_1}{\partial x_2} \frac{dU_0}{dy_2} - y_2 U_0 \left[(M_\infty^2 - 1) \frac{\partial p_{21}}{\partial x_2} - \frac{\partial^2}{\partial z_2^2} \int_{-\infty}^{x_2} p_{21} d\xi \right] -$$

$$M_\infty^2 U_0 \left[\frac{\partial p_{21}}{\partial x_2} + \frac{\partial^2}{\partial z_2^2} \int_{-\infty}^{x_2} p_{21} d\xi \right] \int_{y_2}^{\infty} \left[\frac{1}{M_0^2(\eta)} - \frac{1}{M_\infty^2} \right] d\eta, \quad \frac{\partial u_{22}}{\partial x_2} + \frac{\partial v_{22}}{\partial y_2} = -M_\infty^2 U_0 \frac{\partial p_{21}}{\partial x_2} + \frac{1}{R_0 U_0} \frac{\partial^2}{\partial z_2^2} \int_{-\infty}^{x_2} p_{21} d\xi$$

$$R_0 U_0 \frac{\partial w_{22}}{\partial x_2} + \frac{\partial p_{22}(t_2, x_2, 0, z_2)}{\partial z_2} = \frac{1}{U_0} \frac{\partial^2}{\partial t_2 \partial z_2} \int_{-\infty}^{x_2} p_{21} d\xi + \left(\frac{1}{U_0} \frac{dU_0}{dy_2} + \frac{1}{R_0} \frac{dR_0}{dy_2} \right) \left[A_1 \frac{\partial p_{21}}{\partial z_2} + \frac{\partial A_1}{\partial x_2} \frac{\partial}{\partial z_2} \int_{-\infty}^{x_2} p_{21} d\xi \right] -$$

$$\frac{\partial^3 A_1}{\partial x_2^2 \partial z_2} \left[y_2 - \int_0^{y_2} \frac{M_\infty^2 - M_0^2(\eta)}{M_\infty^2} d\eta \right]$$

$$U_0 \frac{\partial \rho_{22}}{\partial x_2} + v_{22} \frac{dR_0}{dy_2} = -\frac{\partial A_1}{\partial t_2} \frac{dR_0}{dy_2} + M_\infty^2 R_0 U_0 \frac{\partial p_{21}}{\partial x_2} + A_1 \frac{\partial A_1}{\partial x_2} \left(U_0 \frac{d^2 R_0}{dy_2^2} - \frac{dU_0}{dy_2} \frac{dR_0}{dy_2} \right)$$

where t_2, ξ, z_2 are the arguments of function p_{21} of the integrand, and the arbitrary function $A_2(t_2, x_2, z_2)$ satisfies the condition that $A_2 \to 0$ when $x_2 \to -\infty$, $t_2, z_2 = \mathrm{const}$.

3. The layer next to the wall.

Let us proceed now with the analysis of region 3 where viscosity has the predominant effect on the velocity field structure. In that region it is necessary to set

$$t = L/U_\infty (t_0 + \varepsilon^2 t_3), \quad x = L(1 + \varepsilon^3 x_3), \quad y = \varepsilon^5 L y_3, \quad z = \varepsilon^3 L z_3 \tag{3.1}$$

and write the expansions of gas parameters in the form

$$v_x = U_\infty (\varepsilon u_{31} + \varepsilon^2 u_{32} + \ldots), \; v_y = U_\infty (\varepsilon^3 v_{31} + \varepsilon^4 v_{32} + \ldots), \quad v_z = U_\infty (\varepsilon w_{31} + \varepsilon^2 w_{32} + \ldots), \tag{3.2}$$

$$\rho = \rho_\infty (\rho_{31} + \varepsilon \rho_{32} + \ldots), \quad p = p_\infty + \rho_\infty U_\infty^2 (\varepsilon^2 p_{31} + \varepsilon^3 p_{32} + \ldots)$$

where t_3, x_3, y_3, z_3 are the arguments of functions $u_{3i}, v_{3i}, w_{3i}, \rho_{3i}, p_{3i}$ $(i = 1, 2, \ldots)$.

The comparision of formulas (1.1), (2.1), and (3.1) shows that $t_1 = t_2 = t_3$, $x_1 = x_2 = x_3$ and $z_1 = z_2 = z_3$, but $y_1 \neq y_2 \neq y_3$, which is natural, since the characteristic dimensions of all three regions in directions lying in the plane under the stream are the same, and the time count in these is carried out in the same way. As regards the scale in the transverse direction to the plate, these in conformity with basic concepts of the free interaction theory /1—6/ are selected differently. As in region 2, the importance of individual Cartesian coordinates in region 3 is essentially nonequitable.

For the derivation of equations in the layer next to the wall it is necessary to write also the expansion for the temperature

$$T = T_\infty (T_{31} + \varepsilon T_{32} + \ldots), \; T_{3i} = T_{3i}(t_3, x_3, y_3, z_3), \quad i = 1, 2, \ldots \tag{3.3}$$

The Clapeyron equation of state $p = R^{(g)} \rho T$ where $R^{(g)}$ is the gas constant, makes it possible to eliminate from the analysis functions T_{31} and T_{32} by expressing them in terms of quantities appearing only in the expansions of density. These expressions can be used by examining the specific heat at constant pressure c_p and the coefficients of first viscosity $\lambda^{(g)}$ and thermal conductivity k, which are usually assumed dependent on one temperature. For shortening calculations it is convenient to introduce the ratios $q = \rho / c_p$ and $\chi = k / \rho^2$. For the indicated thermodynamic functions the following expansions:

$$\lambda^{(g)} = \lambda_\infty^{(g)} (\lambda_{31}^{(g)} + \varepsilon \lambda_{32}^{(g)} + \ldots), \quad q = \frac{\rho_\infty}{c_{p\infty}} (q_{31} + \varepsilon q_{32} + \ldots), \quad \chi = \frac{k_\infty}{\rho_\infty^2} (\chi_{31} + \varepsilon \chi_{32} + \ldots) \tag{3.4}$$

where t_3, x_3, y_3, z_3 are the arguments of functions $\lambda^{(g)}_{3i}, q_{3i}, \chi_{3i}$ $(i = 1, 2, \ldots)$ are valid.

Introducing formulas (3.1) together with the asymptotic sequencies (3.2)—(3.4) in the system of Navier—Stokes equations, we obtain the usual Prandtl equations

$$\frac{\partial \rho_{31}}{\partial t_3}+\frac{\partial \rho_{31}u_{31}}{\partial x_3}+\frac{\partial \rho_{31}v_{31}}{\partial y_3}+\frac{\partial \rho_{31}w_{31}}{\partial z_3}=0 \tag{3.5}$$

$$\rho_{31}\left(\frac{\partial u_{31}}{\partial t_3}+u_{31}\frac{\partial u_{31}}{\partial x_3}+v_{31}\frac{\partial u_{31}}{\partial y_3}+w_{31}\frac{\partial u_{31}}{\partial z_3}\right)=-\frac{\partial p_{31}}{\partial x_3}+\frac{\partial}{\partial y_3}\left(\lambda_{31}^{(g)}\frac{\partial u_{31}}{\partial y_3}\right), \qquad \frac{\partial p_{31}}{\partial y_3}=0$$

$$\rho_{31}\left(\frac{\partial w_{31}}{\partial t_3}+u_{31}\frac{\partial w_{31}}{\partial x_3}+v_{31}\frac{\partial w_{31}}{\partial y_3}+w_{31}\frac{\partial w_{31}}{\partial z_3}\right)=-\frac{\partial p_{31}}{\partial z_3}+\frac{\partial}{\partial y_3}\left(\lambda_{31}^{(g)}\frac{\partial w_{31}}{\partial y_3}\right)$$

$$\frac{\partial \rho_{31}}{\partial t_3}+u_{31}\frac{\partial \rho_{31}}{\partial x_3}+v_{31}\frac{\partial \rho_{31}}{\partial y_3}+w_{31}\frac{\partial \rho_{31}}{\partial z_3}=\frac{1}{\Pr}q_{31}\frac{\partial}{\partial y_3}\left(\chi_{31}\frac{\partial \rho_{31}}{\partial y_3}\right)$$

for the unsteady three-dimensional boundary layer in an incompressible gas which are satisfied by the principal terms. The difference is, however, in that the perturbation pressure cannot be taken from the solution of the external flow problem. In the considered here boundary layer both derivatives $\partial p_{31}/\partial x_3$ and $\partial p_{31}/\partial z_3$ are nonzero. Equations (3.5) must be supplemented by the final relations between the thermodynamic coefficients $\lambda_{31}^{(g)}$, q_{31} and χ_{31} and perturbations of density ρ_{31} and pressure p_{31}. As usual, the Prandtl number $\Pr$ is taken as the ratio of the Péclet and the Reynolds numbers, i.e. $\Pr = c_{p_\infty}\lambda_\infty^{(g)}/k_\infty$.

For the second approximation functions we have

$$\frac{\partial \rho_{32}}{\partial t_3}+\frac{\partial(\rho_{31}u_{32}+\rho_{32}u_{31})}{\partial x_3}+\frac{\partial(\rho_{31}v_{32}+\rho_{32}v_{31})}{\partial y_3}+\frac{\partial(\rho_{31}w_{32}+\rho_{32}w_{31})}{\partial z_3}=0 \tag{3.6}$$

$$\begin{aligned}&\rho_{31}\frac{\partial u_{32}}{\partial t_3}+\rho_{32}\frac{\partial u_{31}}{\partial t_3}+\rho_{31}u_{31}\frac{\partial u_{32}}{\partial x_3}+(\rho_{31}u_{32}+\rho_{32}u_{31})\frac{\partial u_{31}}{\partial x_3}+\\ &\quad\rho_{31}v_{31}\frac{\partial u_{32}}{\partial y_3}+(\rho_{31}v_{32}+\rho_{32}v_{31})\frac{\partial u_{31}}{\partial y_3}+\rho_{31}w_{31}\frac{\partial u_{32}}{\partial z_3}+\\ &\quad(\rho_{31}w_{32}+\rho_{32}w_{31})\frac{\partial u_{31}}{\partial z_3}=-\frac{\partial p_{32}}{\partial x_3}+\frac{\partial}{\partial y_3}\left(\lambda_{31}^{(g)}\frac{\partial u_{32}}{\partial y_3}+\lambda_{32}^{(g)}\frac{\partial u_{31}}{\partial y_3}\right)\end{aligned}$$

$$\begin{aligned}&\frac{\partial p_{32}}{\partial y_3}=0\\ &\rho_{31}\frac{\partial w_{32}}{\partial t_3}+\rho_{32}\frac{\partial w_{31}}{\partial t_3}+\rho_{31}u_{31}\frac{\partial w_{32}}{\partial x_3}+(\rho_{31}u_{32}+\rho_{32}u_{31})\frac{\partial w_{31}}{\partial x_3}+\\ &\quad\rho_{31}v_{31}\frac{\partial w_{32}}{\partial y_3}+(\rho_{31}v_{32}+\rho_{32}v_{31})\frac{\partial w_{31}}{\partial y_3}+\rho_{31}w_{31}\frac{\partial w_{32}}{\partial z_3}+\\ &\quad(\rho_{31}w_{32}+\rho_{32}w_{31})\frac{\partial w_{31}}{\partial z_3}=-\frac{\partial p_{32}}{\partial z_3}+\frac{\partial}{\partial y_3}\left(\lambda_{31}^{(g)}\frac{\partial w_{32}}{\partial y_3}+\lambda_{32}^{(g)}\frac{\partial w_{31}}{\partial y_3}\right)\\ &\frac{\partial \rho_{32}}{\partial t_3}+u_{32}\frac{\partial \rho_{31}}{\partial x_3}+u_{31}\frac{\partial \rho_{32}}{\partial x_3}+v_{32}\frac{\partial \rho_{31}}{\partial y_3}+v_{31}\frac{\partial \rho_{32}}{\partial y_3}+w_{32}\frac{\partial \rho_{31}}{\partial z_3}+\\ &\quad w_{31}\frac{\partial \rho_{32}}{\partial z_3}=\frac{1}{\Pr}\left[q_{31}\frac{\partial}{\partial y_3}\left(\chi_{31}\frac{\partial \rho_{32}}{\partial y_3}+\chi_{32}\frac{\partial \rho_{31}}{\partial y_3}\right)+q_{32}\frac{\partial}{\partial y_3}\left(\chi_{31}\frac{\partial \rho_{31}}{\partial y_3}\right)\right]\end{aligned}$$

which is nothing else but the linearized Prandtl equations for unsteady three-dimensional flows of compressible gas. The remaining terms in the input system of Navier—Stokes equations affect only the construction of higher approximations. The homogeneity of all equations of system (3.6) is related to the latter feature. In that system the thermodynamic quantities $\lambda_{31}^{(g)}, \lambda_{32}^{(g)}, q_{31}, q_{32}, \chi_{31}$, χ_{32} are to be expressed in terms of perturbations of density ρ_{31}, ρ_{32} and pressure p_{31}, p_{32}, which is achieved by the preliminary substitution of formula (3.3) for temperature into the Clapeyron equation of state.

4. Merging of asymptotic expansions. To effect the merging of the considered asymptotic sequencies it is necessary to know the behavior of solution when approaching from inside the upper and lower boundaries of region 2. Since $R_0(y_2)\to 1$ and $U_0(y_2)\to 1$, as $y_2\to\infty$, formulas (2.6) yield

$$p_{22}-y_2\frac{\partial^2 A_1}{\partial x_2^2}\to p_{22}(t_2,x_2,0,z_2)-\frac{\partial^2 A_1}{\partial x_2^2}\int_0^\infty\frac{M_\infty^2-M_0^2(\eta)}{M_\infty^2}d\eta \tag{4.1}$$

$$v_{22}+y_2\left[(M_\infty^2-1)\frac{\partial p_{21}}{\partial x_2}-\frac{\partial^2}{\partial z_2^2}\int_{-\infty}^{x_2}p_{21}(t_2,\xi,z_2)\,d\xi\right]\to-\frac{\partial A_1}{\partial t_2}-\frac{\partial A_2}{\partial x_2}$$

$$w_{22}+y_2\frac{\partial^2 A_1}{\partial x_2\partial z_2}\to\frac{\partial^2}{\partial t_2\partial z_2}\int_{-\infty}^{x_2}p_{21}(t_2,\xi,z_2)\,d\xi-\frac{\partial}{\partial z_2}\int_{-\infty}^{x_2}p_{22}(t_2,\xi,0,z_2)\,d\xi+\frac{\partial^2 A_1}{\partial x_2\partial z_2}\int_0^\infty\frac{M_\infty^2-M_0^2(\eta)}{M_\infty^2}d\eta$$

$$u_{22}\to -p_{21}(t_2,x_2,z_2),\quad \rho_{22}\to M_\infty^2 p_{21}(t_2,x_2,z_2)$$

These formulas are valid for any conditions at the plate. The behavior of solution near the lower boundary of region 2 depends on thermal conditions maintained at the surface in the stream. We assume for simplicity that the plate is thermally insulated. Denoting by $\varkappa$ the ratio of specific heats, we find that the relations

$$\frac{dR_0^\circ}{dy_2}=0,\quad \frac{d^2R_0^\circ}{dy_2^2}=(\varkappa-1)M_\infty^2\,\mathrm{Pr}\left[R_0^\circ\frac{dU_0^\circ}{dy_2}\right]^2,\quad \frac{d^2U_0^\circ}{dy_2^2}=0,\qquad (R_0^\circ=R_0(0),\ U_0^\circ=U_0(0))$$

which follow from the Blasius solution /12/ hold at the plate surface. Taking these into account it is possible to show that as $y_2\to 0$ the functions

$$p_{22}\to p_{22}(t_2,x_2,0,z_2) \tag{4.2}$$

$$v_{22}\to-\frac{\partial A_1}{\partial t_2}-A_1\frac{\partial A_1}{\partial x_2}-\left[R_0^\circ\frac{dU_0^\circ}{dy_2}\right]^{-1}\left[\frac{\partial p_{21}}{\partial x_2}+\frac{\partial^2}{\partial z_2^2}\int_{-\infty}^{x_2}p_{21}\,d\xi\right]$$

$$u_{22}\to\frac{dU_0^\circ}{dy_2}A_2+\left[y_2R_0^\circ\frac{dU_0^\circ}{dy_2}\right]^{-1}\frac{\partial^2}{\partial z^2}\int_{-\infty}^{x_2}\int_{-\infty}^{\zeta}p_{21}\,d\xi\,d\zeta+$$

$$M_\infty^2\frac{dU_0^\circ}{dy_2}\left[p_{21}+\frac{\partial^2}{\partial z_2^2}\int_{-\infty}^{x_2}\int_{-\infty}^{\zeta}p_{21}\,d\xi\,d\zeta\right]\times\int_0^{\infty}\left\{\frac{1}{M_0^2(\eta)}-\left[\eta\frac{dM_0^\circ}{dy_2}\right]^{-2}-\frac{1}{M_\infty^2}\right\}d\eta$$

$$w_{22}\to\left[y_2^2R_0^\circ\frac{dU_0^\circ}{dy_2}\right]^{-1}\int_{-\infty}^{x_2}\left\{A_1\frac{\partial p_{21}}{\partial z_2}+\frac{\partial A_1}{\partial\zeta}\frac{\partial}{\partial z_2}\int_{-\infty}^{\zeta}p_{21}\,d\xi+\right.$$

$$\left.\left[\frac{dU_0^\circ}{dy_2}\right]^{-1}\frac{\partial^2}{\partial t_2\partial z_2}\int_{-\infty}^{\zeta}p_{21}\,d\xi\right\}d\zeta-\left[y_2R_0^\circ\frac{dU_0^\circ}{dy_2}\right]^{-1}\frac{\partial}{\partial z_2}\int_{-\infty}^{x_2}p_{22}(t_2,\xi,0,z_2)\,d\xi+$$

$$\left[(R_0^\circ)^2\frac{dU_0^\circ}{dy_2}\right]^{-1}\frac{d^2R_0^\circ}{dy_2^2}\int_{-\infty}^{x_2}\left[A_1\frac{\partial p_{21}}{\partial z_2}+\frac{\partial A_1}{\partial\zeta}\frac{\partial}{\partial z_2}\int_{-\infty}^{\zeta}p_{21}\,d\xi\right]d\zeta$$

$$\rho_{22}\to M_\infty^2R_0^\circ p_{21}+\frac{d^2R_0^\circ}{dy_2^2}\left\{\frac{1}{2}A_1^2+\frac{1}{R_0^\circ}\left[\frac{dU_0^\circ}{dy_2}\right]^{-2}\times\left[p_{21}+\frac{\partial^2}{\partial z_2^2}\int_{-\infty}^{x_2}\int_{-\infty}^{\zeta}p_{21}\,d\xi\,d\zeta\right]\right\},\quad M_0^\circ=M_0(0)$$

where t_2,ξ,z_2 and t_2,ζ,z_2 are the arguments of functions p_{21} and A_1, respectively, of the integrands.

It should be noted that in the expression for u_{22} there is a singularity of the form y_2^{-1}, and in the asymptotic formula for w_{22} besides a singularity of order y_2^{-1} there is, also, a stronger singularity of the form y_2^{-2}. These singularities are inherent to three-dimensional flows, and are absent in respective expansions for the plane-parallel boundary layer with $\partial/\partial z_2=0$.

Using the formulas (2.5) for the principal terms of sequencies and the asymptotic expressions (4.1) for the functions of the second approximation in order to obtain the boundary conditions, which must be satisfied in the construction of solution in region 1. When $y_2\to\infty$, then the external variable $y_1\to 0$. This implies that

$$p_{11}(t_1,x_1,0,z_1)=p_{21}(t_2,x_2,z_2),\quad v_{11}(t_1,x_1,0,z_1)=-\frac{\partial A_1}{\partial x_2} \tag{4.3}$$

$$u_{11}(t_1,x_1,0,z_1)=-p_{21}(t_2,x_2,z_2),\quad w_{11}(t_1,x_1,0,z_1)=-\frac{\partial}{\partial z_2}\int_{-\infty}^{x_2}p_{21}(t_2,\xi,z_2)\,d\xi$$

$$\rho_{11}(t_1,x_1,0,z_1)=M_\infty^2p_{21}(t_2,x_2,z_2)$$

where only the first two conditions are independent, the remaining exactly coincide with the second, third, and fifth equations of system (1.5), and are to be defined in the plane $y_1=0$.

Moreover, the perturbations of pressure, transverse and lateral components of the velocity vector we have in the second approximation the relations

$$p_{12}(t_1,x_1,0,z_1)=p_{22}(t_2,x_2,0,z_2)-\frac{\partial^2A_1}{\partial x_2^2}\int_0^{\infty}\frac{M_\infty^2-M_0^2(\eta)}{M_\infty^2}\,d\eta,\quad v_{12}(t_1,x_1,0,z_1)=-\frac{\partial A_1}{\partial t_2}-\frac{\partial A_2}{\partial x_2} \tag{4.4}$$

$$w_{12}(t_1,x_1,0,z_1)=-\frac{\partial}{\partial z_2}\int_{-\infty}^{x_2}p_{22}(t_2,\xi,0,z_2)\,d\xi+\frac{\partial^2A_1}{\partial x_2\partial z_2}\int_0^{\infty}\frac{M_\infty^2-M_0^2(\eta)}{M_\infty^2}\,d\eta+\frac{\partial^2}{\partial t_2\partial z_2}\int_{-\infty}^{x_2}\int_{-\infty}^{\zeta}p_{21}(t_2,\xi,z_2)\,d\xi\,d\zeta$$

of which, again, only the first two are independent. The last of conditions (4.4) is readily reduced to the form

$$w_{12}(t_1, x_1, 0, z_1) = -\frac{\partial}{\partial z_1}\int_{-\infty}^{x_1} p_{12}(t_1, \xi, 0, z_1)\,d\xi - \frac{\partial^2}{\partial t_1 \partial z_1}\int_{-\infty}^{x_1}\int_{-\infty}^{\zeta} p_{11}(t_1, \xi, 0, z_1)\,d\xi\,d\zeta$$

which follows from the fifth of Eqs.(1.8) considered in the plane $y_1 = 0$. The boundary conditions for perturbations of density and the velocity vector longitudinal component cannot be derived in the second approximation for $y_1 = 0$, using the asymptotic expressions (4.1). To do this it is necessary to know the third approximation terms in the solution for region 2, which are not considered in the present analysis. For plane-parallel motions of gas in the $y_1 = 0$ plane we have formulas (1.6) or (1.7) depending on whether the Mach number at infinity exceeds unity or remains below it. Taking into account the above relations, we come to the immediate conclusion that the boundary conditions for $y_1 = 0$ for the principal terms of solution in region 1 can be expressed in terms of function $A_1(t_2, x_2) = A_1(t_1, x_1)$ if $\partial/\partial z_1 = \partial/\partial z_2 = 0$. In that case the boundary conditions for second approximation terms will also contain function $A_2(t_2, x_2) = A_2(t_1, x_1)$. If, however, the velocity field in the boundary layer has a three-dimensional structure, then in the relation

$$v_{11}(t_1, x_1, 0, z_1) = -\int_{-\infty}^{x_1}\frac{\partial p_{11}(t_1, \xi, 0, z_1)}{\partial y_1}\,d\xi$$

it is not possible to get rid of the normal derivative of the excess pressure by exchanging it for the function itself.

Let us now carry out the merging of expansions which represent the asymptotic form of solutions in regions 2 and 3. Reverting to formulas (2.5) and expressions (4.2), we obtain limit conditions which are to be satisfied by the parameters of gas in the thin layer next to the wall. If $y_2 \to 0$, then the internal variable $y_3 \to \infty$ and the sought quantities are

$$p_{31}(t_3, x_3, z_3) \to p_{21}(t_2, x_2, z_2),\ \rho_{31} \to R_0^\circ, \quad u_{31} - y_3\frac{dU_0^\circ}{dy_2} \to \frac{dU_0^\circ}{dy_2}A_1 + \left[y_3 R_0^\circ \frac{dU_0^\circ}{dy_2}\right]^{-1}\frac{\partial^2}{\partial z_2^2}\int_{-\infty}^{x_2}\int_{-\infty}^{\zeta} p_{21}\,d\xi\,d\zeta \qquad (4.5)$$

$$w_{31} \to -\left[y_3 R_0^\circ\frac{dU_0^\circ}{dy_2}\right]^{-1}\frac{\partial}{\partial z_2}\int_{-\infty}^{x_2} p_{21}\,d\xi + \left[y_3^2 R_0^\circ\frac{dU_0^\circ}{dy_2}\right]^{-1}\int_{-\infty}^{x_2}\left\{A_1(t_2, \zeta, z_2)\frac{\partial p_{21}}{\partial z_2} + \frac{\partial A_1}{\partial \zeta}\frac{\partial}{\partial z_2}\int_{-\infty}^{\zeta} p_{21}\,d\xi + \right.$$

$$\left.\left[\frac{dU_0^\circ}{dy_2}\right]^{-1}\frac{\partial^2}{\partial t_2 \partial z_2}\int_{-\infty}^{\zeta} p_{21}\,d\xi\right\}d\zeta$$

where t_2, ξ, z_2 are the arguments of function p_{21} in the integrand.

The limit condition for the velocity vector component along the normal to the body surface is usually omitted. In this case it can be written in the form

$$v_{31} + y_3\frac{dU_0^\circ}{dy_2}\frac{\partial A_1}{\partial x_2} \to -\frac{\partial A_1}{\partial t_2} - A_1\frac{\partial A_1}{\partial x_2}\frac{dU_0^\circ}{dy_2} - \left[R_0^\circ\frac{dU_0^\circ}{dy_2}\right]^{-1}\left[\frac{\partial p_{21}}{\partial x_2} + \frac{\partial^2}{\partial z_2^2}\int_{-\infty}^{x_2} p_{21}(t_2, \xi, z_2)\,d\xi\right]$$

and is automatically satisfied, when conditions (4.5) for pressure perturbations, density, and velocity vector components lying in a plane tangent to the body surface are satisfied. Indeed, by substituting the asymptotics of all first approximation functions which determine the structure of the viscous layer next to the wall, into Eqs.(3.5), we can verify the validity of the above statement, since this substitution results in a system of identities.

The boundary conditions for second approximation functions are

$$p_{32}(t_3, x_3, z_3) \to p_{22}(t_2, x_2, 0, z_2),\ \rho_{32} \to 0 \qquad (4.6)$$

$$u_{32} \to \frac{dU_0^\circ}{dy_2}\left\{A_2 + M_\infty^2\left[p_{21} + \frac{\partial}{\partial z_2^2}\int_{-\infty}^{x_2}\int_{-\infty}^{\zeta} p_{21}(t_2, \xi, z_2)\,d\xi\,d\zeta\right] \times \int_0^{\infty}\left[\frac{1}{M_0^2(\eta)} - \left[\eta\frac{dM_0(0)}{dy_2}\right]^{-2} - \frac{1}{M_\infty^2}\right]d\eta\right\}$$

$$w_{32} \to -\left[y_2 R_0^\circ\frac{dU_0^\circ}{dy_2}\right]^{-1}\frac{\partial}{\partial z_2}\int_{-\infty}^{x_2} p_{22}(t_2, \xi, 0, z_2)\,d\xi$$

Inserting into the second of equalities (4.2) for v_{22} the supplementary term proportional to y_2, we obtain the limit condition

$$v_{32} \to -y_3\frac{dU_0^\circ}{dy_2}\left\{\frac{\partial A_2}{\partial x_2} + M_\infty^2\left[\frac{\partial p_{21}}{\partial x_2} + \frac{\partial^2}{\partial z_2^2}\int_{-\infty}^{x_2} p_{21}(t_2, \xi, z_2)\,d\xi\right] \times \int_0^{\infty}\left[\frac{1}{M_0^2(\eta)} - \left[\eta\frac{dM_0^\circ}{dy_2}\right]^{-2} - \frac{1}{M_\infty^2}\right]d\eta\right\}$$

for the transverse velocity vector component. As in the first approximation, it is automatically satisfied, if the conditions for perturbations of pressure, density, and velocity vector components in the plane tangent to the body surface are assumed satisfied. This can be readily proved by substituting the asymptotic expressions for the quantities $p_{32}, \rho_{32}, u_{32}, v_{32}, w_{32}$ together with similar expressions for functions $p_{31}, \rho_{31}, u_{31}, v_{31}, w_{31}$ into the linear equations (3.6), since each of them becomes an identity.

Thus the boundary condition for the velocity vector component normal to the plate can be rejected, when integrating the system of equations for the viscous layer next to the wall. Formally this appears to be exactly as in the classical Prandtl theory. However the reasons for the omission of that boundary condition are exactly opposite in the two cases, viz., when the self-induced pressure is taken into account it follows from the remaining boundary conditions imposed as $y_3 \to \infty$, while in the conventional theory of the boundary layer it is superfluous making the respective boundary value problem insoluble.

Note that the term ρ_{22} in the expansion for density was not taken at all into consideration in the process of merging the solutions for regions 2 and 3. This is reasonable, since when $y_2 \to 0$ the contribution due to it is proportional to ε^2, it is sufficient to specify the gas density throughout the thin layer next to the wall with an accuracy to terms of order ε. As regards terms with w_{22} in the expansion of the velocity vector lateral component, its singular values proportional to y_2^{-2} and y_2^{-1} were used in the merging with the solution in region 3, and only the regular residual yielding a contribution of order ε^3 was omitted.

5. Boundary value problems. Below, we assume that the specific heat at constant pressure is constant and the coefficients of viscosity and thermal conductivity conform to Chapman's linear laws

$$\lambda^{(s)}/\lambda_\infty^{(s)} = cT/T_\infty, \quad k/k_\infty = cT/T_\infty, \quad c = \text{const}$$

and set the Prandtl number equal unity. Then the ratio T_w/T_∞ of wall and oncoming stream temperatures is obtained from the Crocco relation /12/

$$T_w/T_\infty = 1 + (\varkappa - 1) M_\infty^2/2$$

As previously indicated, in the case of a thermally insulated plate the derivative $dR_0^\circ/dy_2 = 0$. From this we conclude that $\rho/\rho_\infty \to R_0^\circ$ as $x \to -\infty$ not only in the first but also in the second approximation. Similarly $\rho/\rho_\infty \to R_0^\circ$ as $y_3 \to +\infty$ in the second formulas of (4.5) and (4.6) obtained by merging solutions for regions 2 and 3. Hence we take as the solution

$$\rho_{31}(t_3, x_3, y_3, z_3) = R_0^\circ, \quad \rho_{32}(t_3, x_3, y_3, z_3) = 0, \quad p_{31}(t_3, x_3, z_3) = p_{21}(t_2, x_2, z_2), \quad p_{32}(t_3, x_3, z_3) = p_{22}(t_2, x_2, 0, z_2)$$

In the considered here case the last of equations appearing in systems (3.5) and (3.6) become identities. Then for first approximation functions in region 3 we have

$$\frac{\partial u_{31}}{\partial x_3} + \frac{\partial v_{31}}{\partial y_3} + \frac{\partial w_{31}}{\partial z_3} = 0, \quad \frac{\partial p_{31}}{\partial y_3} = 0 \tag{5.1}$$
$$R_0^\circ\left(\frac{\partial u_{31}}{\partial t_3} + u_{31}\frac{\partial u_{31}}{\partial x_3} + v_{31}\frac{\partial u_{31}}{\partial y_3} + w_{31}\frac{\partial u_{31}}{\partial z_3}\right) = -\frac{\partial p_{31}}{\partial x_3} + \frac{c}{R_0^\circ}\frac{\partial^2 u_{31}}{\partial y_3^2}$$
$$R_0^\circ\left(\frac{\partial w_{31}}{\partial t_3} + u_{31}\frac{\partial w_{31}}{\partial x_3} + v_{31}\frac{\partial w_{31}}{\partial y_3} + w_{31}\frac{\partial w_{31}}{\partial z_3}\right) = -\frac{\partial p_{31}}{\partial z_3} + \frac{c}{R_0^\circ}\frac{\partial^2 w_{31}}{\partial y_3^2}$$

and second approximation functions are determined using the system

$$\frac{\partial u_{32}}{\partial x_3} + \frac{\partial v_{32}}{\partial y_3} + \frac{\partial w_{32}}{\partial z_3} = 0, \quad \frac{\partial p_{32}}{\partial y_3} = 0 \tag{5.2}$$

$$R_0^\circ\left(\frac{\partial u_{32}}{\partial t_3} + u_{31}\frac{\partial u_{32}}{\partial x_3} + u_{32}\frac{\partial u_{31}}{\partial x_3} + v_{31}\frac{\partial u_{32}}{\partial y_3} + v_{32}\frac{\partial u_{31}}{\partial y_3} + w_{31}\frac{\partial u_{32}}{\partial z_3} + w_{32}\frac{\partial u_{31}}{\partial z_3}\right) = -\frac{\partial p_{32}}{\partial x_3} + \frac{c}{R_0^\circ}\frac{\partial^2 u_{32}}{\partial y_3^2}$$

$$R_0^\circ\left(\frac{\partial w_{32}}{\partial t_3} + u_{31}\frac{\partial w_{32}}{\partial x_3} + u_{32}\frac{\partial w_{31}}{\partial x_3} + v_{31}\frac{\partial w_{32}}{\partial y_3} + v_{32}\frac{\partial w_{31}}{\partial y_3} + w_{31}\frac{\partial w_{32}}{\partial z_3} + w_{32}\frac{\partial w_{31}}{\partial z_3}\right) = -\frac{\partial p_{32}}{\partial z_3} + \frac{c}{R_0^\circ}\frac{\partial^2 w_{32}}{\partial y_3^2}$$

System (5.1) is formed of usual Prandtl equations for an unsteady three-dimensional boundary layer in an incompressible fluid. System (5.2) consists of Prandtl linearized equations which define unsteady three-dimensional incompressible flows. In both systems pressure perturbations p_{31} and p_{32} must be determined and are not obtained from solutions of the external flow, hence for a plate $p_{31} \neq 0$ and $p_{32} \neq 0$.

Let us now carry out the similarity transformation

$$|M_\infty^2 - 1| = \delta, \quad T_w / T_\infty = T_0 \tag{5.3}$$
$$t_1 = t_2 = t_3 = c^{1/2}\lambda^{-2}\delta^{-1/4}T_0 t''$$
$$x_1 = x_2 = x_3 = c^{3/4}\lambda^{-5/4}\delta^{-3/8}T_0^{3/2} x''$$

$$y_1 = c^{1/4}\lambda^{-3/4}\delta^{-1}T_0^{3/2}y', \quad y_3 = c^{1/4}\lambda^{-5/4}\delta^{-1/4}T_0^{3/2}y''$$
$$z_1 = z_2 = z_3 = c^{3/8}\lambda^{-5/4}\delta^{-3/4}T_0^{3/2}z''$$
$$u_{31} + \varepsilon u_{32} = c^{1/8}\lambda^{1/4}\delta^{-1/4}T_0^{1/2}(u_{31}'' + \varepsilon u_{32}'')$$
$$v_{11} + \varepsilon v_{12} = c^{1/4}\lambda^{1/2}\delta^{-1/4}(v_{11}' + \varepsilon v_{12}'), \quad v_{31} + \varepsilon v_{32} = c^{3/8}\lambda^{3/4}\delta^{1/4}T_0^{1/2}(v_{31}'' + \varepsilon v_{32}'')$$
$$w_{31} + \varepsilon w_{32} = c^{1/8}\lambda^{1/4}\delta^{-1/4}T_0^{1/2}(w_{31}'' + \varepsilon w_{32}''), \quad p_{11} + \varepsilon p_{12} = c^{1/4}\lambda^{1/2}\delta^{-1/4}(p_{11}' + \varepsilon p_{12}')$$
$$p_{21} + \varepsilon p_{22} = c^{1/4}\lambda^{1/2}\delta^{-1/4}(p_{21}^{(m)} + \varepsilon p_{22}^{(m)}), \quad p_{31} + \varepsilon p_{32} = c^{1/4}\lambda^{1/2}\delta^{-1/4}(p_{31}'' + \varepsilon p_{32}'')$$
$$A_1 + \varepsilon A_2 = c^{5/8}\lambda^{-5/4}\delta^{-1/4}T_0^{3/2}(A_1'' + \varepsilon A_2'')$$

where the constant $\lambda = 0{,}3321$ is determined using $dU_0^\circ/dy_2 = \lambda c^{-1/2}T_0^{-1}$ and is calculated using the Blasius solution of an unperturbed boundary layer. This transformation enables us to exclude from the formulation of the problem the dependence of principal terms of sequencies on constants c and $R_0^\circ = T_0^{-1}$, while the remainder $M_\infty^2 - 1$ appears only in relations for the external region of the stream.

Let us begin by formulating the boundary value problem for first approximation functions. The first and fourth equations of system (1.5), expressed in new variables, at which for simplicity double primes have been omitted, assume the canonical form

$$\mp\frac{\partial^2 p_{11}'}{\partial x^2} + \frac{\partial^2 p_{11}'}{\partial y'^2} + |M_\infty^2 - 1|^{-1}\frac{\partial^2 p_{11}'}{\partial z^2} = 0, \quad v_{11}' = -|M_\infty^2 - 1|^{-1/2}\frac{\partial}{\partial y'}\int_{-\infty}^{x} p_{11}'(t, \xi, y', z)\, d\xi \tag{5.4}$$

where the upper sign at the derivative $\partial^2 p_{11}'/\partial x^2$ applies when the oncoming stream is supersonic, and the lower when it is subsonic at infinity.

The boundary condition for $y' = 0$ are derived from the first two of equalities (4.3)

$$p_{11}' = p_{21}^{(m)}(t, x, z), \quad v_{11}' = -|M_\infty^2 - 1|^{1/2}\frac{\partial A_1}{\partial x} \tag{5.5}$$

which are independent. As previously noted, the remaining relation of (4.3) follow from respective equations of system (1.5) considered in the $y' = 0$ plane. It is, consequently, possible to omit all supplementary equations and boundary conditions when formulating the boundary value problem for $y' = 0$.

The remaining boundary conditions are formulated as limit conditions. Namely, as $x \to -\infty$ and $y' \to +\infty$ we have

$$p_{11}' \to 0, \quad v_{11}' \to 0 \tag{5.6}$$

Substituting expressions (5.3) into system (5.1) and omitting the double primes at the newly introduced variables, we obtain equation in the canonical form

$$\frac{\partial u_{31}}{\partial x} + \frac{\partial v_{31}}{\partial y} + \frac{\partial w_{31}}{\partial z} = 0, \quad \frac{\partial p_{31}}{\partial y} = 0, \quad \frac{\partial u_{31}}{\partial t} + u_{31}\frac{\partial u_{31}}{\partial x} + v_{31}\frac{\partial u_{31}}{\partial y} + w_{31}\frac{\partial u_{31}}{\partial z} = -\frac{\partial p_{31}}{\partial x} + \frac{\partial^2 u_{31}}{\partial y^2} \tag{5.7}$$

$$\frac{\partial w_{31}}{\partial t} + u_{31}\frac{\partial w_{31}}{\partial x} + v_{31}\frac{\partial w_{31}}{\partial y} + w_{31}\frac{\partial w_{31}}{\partial z} = -\frac{\partial p_{31}}{\partial z} + \frac{\partial^2 w_{31}}{\partial y^2}$$

for which we have the obvious boundary conditions when $y = 0$

$$u_{31} = 0, \quad v_{31} = 0, \quad w_{31} = 0 \tag{5.8}$$

The remaining boundary conditions are formulated here also as limiting. As $x \to -\infty$

$$u_{31} \to y, \quad w_{31} \to 0, \quad p_{31} \to 0 \tag{5.9}$$

Moreover, on the basis of formulas (4.5) we conclude that as $y \to \infty$

$$p_{31} \to p_{21}^{(m)}(t, x, z), \quad u_{31} - y \to A_1(t, x, z) + \frac{1}{y}\frac{\partial^2}{\partial z^2} \times \int_{-\infty}^{x}\int_{-\infty}^{\zeta} p_{21}^{(m)}(t, \xi, z)\, d\xi\, d\zeta \tag{5.10}$$

$$w_{31} \to -\frac{1}{y}\frac{\partial}{\partial z}\int_{-\infty}^{x} p_{21}^{(m)}(t, \xi, z)\, d\xi + \frac{1}{y^2}\int_{-\infty}^{x}\left[A_1(t, \zeta, z)\frac{\partial p_{21}^{(m)}}{\partial z} + \frac{\partial A_1}{\partial \zeta}\frac{\partial}{\partial z}\int_{-\infty}^{\zeta} p_{21}^{(m)}(t, \xi, z)\, d\xi + \frac{\partial^2}{\partial t \partial z}\int_{-\infty}^{x} p_{21}^{(m)}(t, \xi, z)\, d\xi\right] d\zeta$$

Systems (5.4) and (5.7) must be integrated jointly. Linking of their solutions is achieved with the use of the arbitrary functions $p_{21}^{(m)}(t, x, z)$ and $A_1(t, x, z)$ which appear in boundary conditions (5.5) and (5.10), and are to be determined.

Attention had been already drawn to the fact of the impossibility of formulation of boundary conditions for a three-dimensional stream in terms of function $A_1(t, x, z)$ only. The unavoidable consequence of the absence of simple expressions for gas parameters in region 1 is the impossibility of separating in the general problem that of the external flow of gas from the problem of the viscous layer next to the wall. The introduction of time in the theory of steady free interaction of the plane-parallel boundary layer /1—6/ did not result in a fundamental complication of boundary value problems because the equations that define the flow in regions 1 and 2 do not contain in the first approximation derivatives with respect to t. On the contrary, as shown by the above reasoning, boundary value problems on free interaction become considerably more complex when the third space variable is taken into account. As regards the stipulation that perturbations must be damped upstream, its fulfillment is ensured by conditions (5.6) and (5.9).

Transforms (5.3) virtually express the law of similarity for the principal terms of gas parameters in the unsteady three-dimensional boundary layer, since neither the systems of Eqs. (5.4) and (5.7), nor the boundary conditions (5.5), (5.6), and (5.8)—(5.10) contain the quantities $\mathrm{Re}, \varkappa, c, \lambda, T_0$. The same flow modes can, thus, occur at various values of the above constants, and the stream properties are determined by the input data of the problem. It is, however, impossible to eliminate the dependence of the dimensionless characteristics of the three-dimensional boundary layer on the number M_∞. The latter is in complete agreement with the Prandtl—Glauert rule which established the length scale contraction in the lateral direction of the external stream, when the linear approximation is used in its study /13/. The similar contraction of the measurements scale for the region next to the wall is produced differently. The similarity laws for the plane-parallel motion of gas $w = \partial/\partial z = 0$ contain in their formulation also the Mach number at infinity /3,6,11/. In fact the term $|M_\infty^2 - 1|^{1/2}$ is common in the expression which obtains by the substitution of the expression in (5.5) for function V_{11} into the left-hand side of the second of Eqs.(5.4) considered in the plane $y' = 0$.

Let us now turn to second approximation functions. Omitting again the double primes at variables transformed according to formulas (5.3), we write the first and fourth of equations of system (1.8) as

$$\mp \frac{\partial^2 p_{12}'}{\partial x^2} + \frac{\partial^2 p_{12}'}{\partial y'^2} + \delta^{-1} \frac{\partial^2 p_{12}'}{\partial z^2} = 2c^{1/4}\lambda^{5/4} M_\infty^2 \delta^{-1/4} T_0^{1/2} \frac{\partial^2 p_{12}'}{\partial t \partial x} \tag{5.11}$$

$$v_{12}' = -\delta^{5/4} \frac{\partial}{\partial y'} \int_{-\infty}^{x} p_{12}'(t, \xi, y', z)\, d\xi + c^{1/4}\lambda^{1/4}\delta^{3/4} T_0^{1/2} \frac{\partial^2}{\partial t \partial y'} \int_{-\infty}^{x} \int_{-\infty}^{\xi} p_{12}'(t, \xi, y', z)\, d\xi\, d\zeta$$

where the upper sign is for a supersonic oncoming stream and the lower for a subsonic one.

When $y' = 0$ the boundary conditions are obtained from the first two of equalities (4.4). Before writing these down it is useful to calculate the integrals in the asymptotic formulas (4.1) and (4.2). In conformity with the Blasius solution for the boundary layer on a thermally insulated plate we have /12/

$$\int_0^\infty \frac{M_\infty^2 - M_0^2(\eta)}{M_\infty^2}\, d\eta = (2c)^{1/2} T_0 \Delta_1, \quad \Delta_1 = 1.686 \tag{5.12}$$

$$\int_0^\infty \left\{ \frac{1}{M_0^2(\eta)} - \left[\eta \frac{dM_0^2}{dy_2} \right]^{-2} - \frac{1}{M_\infty^2} \right\} d\eta = \overline{\int_0^\infty} \left[\frac{1}{R_0(\eta) U_0^2(\eta)} - 1 \right] d\eta = (2c)^{1/2} [T_0^2 \Delta_2 - T_0 (T_0 - 1) \Delta_1], \quad \Delta_2 = -3.063$$

where the small angle sign over the improper divergent integral denotes it finite part in the Hadamard sense /14/. We have

$$p_{12}' = p_{22}^{(m)}(t, x, 0, z) - 2^{1/4} \Delta_1 c^{1/4} \lambda^{5/4} \delta^{1/4} T_0^{-1/2} \frac{\partial^2 A_1}{\partial x_2}, \quad v_{12}' = -\delta^{1/2} \frac{\partial A_2}{\partial x} - c^{1/4} \lambda^{1/4} \delta^{1/4} T_0^{1/2} \frac{\partial A_1}{\partial t} \tag{5.13}$$

The remaining boundary conditions (4.4) are omitted, since they follow from the respective equations of system (1.8) by setting in it $y' = 0$.

The limit conditions as $x \to -\infty$ and $y' \to +\infty$ are

$$p_{12}' \to 0, \quad v_{12}' \to 0 \tag{5.14}$$

For a viscous flow in the region close to the wall system (5.2) consists of linearized Prandtl equations. It is thus clear that in the transformed variables with omitted double primes they assume the canonical form

$$\frac{\partial u_{32}}{\partial x} + \frac{\partial v_{32}}{\partial y} + \frac{\partial w_{32}}{\partial z} = 0, \quad \frac{\partial p_{32}}{\partial y} = 0 \tag{5.15}$$

$$\frac{\partial u_{32}}{\partial t} + u_{31} \frac{\partial u_{32}}{\partial x} + u_{32} \frac{\partial u_{31}}{\partial x} + v_{31} \frac{\partial u_{32}}{\partial y} + v_{32} \frac{\partial u_{31}}{\partial y} + w_{31} \frac{\partial u_{32}}{\partial z} + w_{32} \frac{\partial u_{31}}{\partial z} = -\frac{\partial p_{32}}{\partial x} + \frac{\partial^2 u_{32}}{\partial y^2}$$

$$\frac{\partial w_{32}}{\partial t} + u_{31} \frac{\partial w_{32}}{\partial x} + u_{32} \frac{\partial w_{31}}{\partial x} + v_{31} \frac{\partial w_{32}}{\partial y} + v_{32} \frac{\partial w_{31}}{\partial y} + w_{31} \frac{\partial w_{32}}{\partial z} + w_{32} \frac{\partial w_{31}}{\partial z} = -\frac{\partial p_{32}}{\partial z} + \frac{\partial^2 w_{32}}{\partial y^2}$$

which is free of any parameters. The boundary conditions of flow $u_{32}=0,\ v_{32}=0,\ w_{32}=0$ when $y=0$ and the limit conditions $u_{32}\to 0,\quad w_{32}\to 0,\ p_{32}\to 0$ as $x\to-\infty$ also do not contain any parameters. It is not so in the case of limit conditions as $y\to\infty$. Taking into consideration the merging requirements (4.6) we conclude that

$$p_{32}\to p_{22}^{(m)}(t,x,0,z) \tag{5.16}$$

$$u_{32}\to A_2(t,x,z)+2^{1/2}c^{1/2}\lambda^{3/4}\delta^{-1/4}T_0^{1/2}[\Delta_2-(1-T_0^{-1})\Delta_1]\times\left[p_{21}^{(m)}(t,x,z)+\frac{\partial^2}{\partial z^2}\int\limits_{-\infty}^{x}\int\limits_{-\infty}^{\zeta}p_{21}^{(m)}(t,\xi,z)\,d\xi\,d\zeta\right]$$

$$w_{32}\to-\frac{1}{y}\frac{\partial}{\partial z}\int\limits_{-\infty}^{x}p_{22}(t,\xi,0,z)\,d\xi$$

where the coefficient Δ_2 is defined by the second of formulas (5.12). The equations of systems (5.11) and (5.15) must be simultaneous integrated. Their solutions are linked by the arbitrary functions $p_{22}^{(m)}(t,x,0,z)$ and $A_2(t,x,z)$ contained in the boundary conditions (5.13) and (5.16). The limit conditions (5.14) must be supplemented by the conditions of damping upstream at infinity of gas parameters in the region next to the wall and the conditions at $y=0$ on the plate. The correction terms of input expansions substantially depend on constants M_∞, c,λ and on T_0, even when these relate to a plane-parallel layer.

We present the following two concluding remarks. First, in the developed theory of free interaction of an unsteady three-dimensional boundary layer with the external stream the characteristic dimensions in all directions on the surface in the stream are of comparable magnitude. This implies that separation zones appear in longitudinal as well as in transverse directions at distances of the order of $\varepsilon^3 L$. When the surface under the boundary layer is rough, separation bubbles of the indicated scale can be generated in the boundary layer region next to such surface. The overall pattern of the stream is then similar to that which occurs at the bottom of a vessel at the beginning of boiling of water in it, although the size of bubbles is evidently different. When the separation is total over the region of the size of characteristic dimensions of the body, the shape of the separation line can substantially vary at distances of the order of $\varepsilon^3 L$.

The second remark relates to the shape of the surface under the stream, which, strictly, speaking, was assumed above to be simply a plane plate. In fact, the surface of the body can be of any shape, provided that the characteristic dimensions remain of the order of L. All reasoning remains valid, except functions $R_0(y)$, $U_0(y)$ and $M_0(y)$ are to be defined not by the self-similar Blasius solution , but by the data obtained by the preliminary integration of Prandtl equations with appropriately formulated boundary conditions. Integration of equations of the boundary layer defines the initial stream on sections of the order of $\varepsilon^3 L$, where the free interaction process takes place. It is for this reason the plane tangent to the body surface was occasionally mentioned instead of the plate.

REFERENCES

1. NEILAND V.Ia., On the theory of the laminar boundary layer separation in a supersonic stream. Izv. Akad. Nauk SSSR, MZhG, No.4, 1969.

2. NEILAND V.Ia., On the asymptotic theory of calculating heat fluxes near a corner point of a body. Izv. Akad. Nauk SSSR, MZhG, No.5, 1969.

3. NEILAND V. Ia., Asymptotic problems of the theory of viscous supersonic flows. Tr. TsAGI, No. 1529. 1974.

4. STEWARTSON K. On the flow near the trailing edge of a flat plate, II. Mathematika, Vol.16, No.31, 1969.

5. STEWARTSON K.and WILLIAMS P.G., Self-induced separation. Proc. Roy. Soc., Ser. A, Vol.312, No. 1509, 1969.

6. STEWARTSON K., Multistructured boundary layers on flat plates and related bodies. In: Advances in Applied Mechanics, Vol.14, New York— San Francisco— London, Acad. Press,1974.

7. SCHNEIDER W., Upstream propagation of unsteady disturbances in supersonic boundary layers. J. Fluid Mech., Vol.63, Pt. 3, 1974.

8. BROWN S.N. and DANIELS P.G., On the viscous flow about the trailing edge of a rapidly oscillating plate. J. Fluid Mech., Vol.67, Pt. 4, 1975.

9. DANIELS P.G., The flow about the trailing edge of a supersonic oscillating aerofoil J. Fluid Mech. Vol.72, Pt.3, 1975.

10. RYZHOV O.S., Equations of the unsteady boundary layer with self-induced pressure. Dokl. Akad.Nauk SSSR, Vol.234, No.4, 1977.

11. RYZHOV O.S. and TERENT'EV E.D., On unsteady boundary layer with self-induced pressure, PMM, Vol.41, No.6, 1977.

12. STEWARTSON K., The Theory of Laminar Boundary Layers in Compressible Fluids. Oxford, Clarendon Press, 1964.

13. LANDAU L.D. and LIFSHITZ, E.M., Mechanics of Continuous Media. Moscow, Gostekhizdat,1954. (See also in English:Course of Theoretical Physics, Pergamon Press).

14. HADAMARD G., The Cauchy Problem for Linear Equations with Partial Derivatives of the Hyperbolic Type. Moscow, "Nauka", 1978.

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